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Hyponormal composition operators on weighted Hardy spaces

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Let $\beta=\{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\beta_0=1$ and $\frac{\beta_{n+1}}{\beta_n}\to 1$ as $n\to\infty$. The set $H^2(\beta)$ of formal complex power series $f(z)=\sum_{n=0}^{\infty} a_n z^n$ such that

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty$$

is a Hilbert space of functions analytic in the unit disc with the inner product

$$(f, g)_\beta = \sum_{n=0}^{\infty} a_n \bar{b}_n \beta_n^2$$

for f as above and $g(z)=\sum_{n=0}^{\infty} b_n z^n$. For details see [9].

If φ is an analytic function mapping the unit disc D into itself, we define the composition operator C_φ on the space $H^2(\beta)$ by $C_\varphi f=f\circ\varphi$. The operators C_φ are not necessarily defined on all of $H^2(\beta)$. They are everywhere defined in some special cases: on the classical Hardy space H^2 (the case when $\beta_n=1$ for all n) — see for example [7], and on a general space $H^2(\beta)$ if the function φ is analytic on some open set containing the closed unit disc having supremum norm strictly smaller than one (see[11]). There are a lot of other known properties of composition operators, on the classical Hardy space H^2 (see for example [1], [6] and [7]), and on more general space $H^2(\beta)$ (see [4], [5], [8], [10] and [11]).

In this article we are interested in the hyponormality of composition operators and their adjoints. The inspiration for this work was COWEN's and KRIETE's article [2] in which, among the other results, they get a nice correlation between hyponormality of composition operators on H^2 and the Denjoy—Wolff point of the inducing map. Their proofs use some properties of the spectrum and spectral radius of a composition operator on H^2 which are still not known in the case of general spaces

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$H^2(\beta)$. Nevertheless, taking a different approach, we can get some results on spaces $H^2(\beta)$.

We say that the operator A on a Hilbert space $\mathscr{H}$ is hyponormal if $A^*A-AA^*\geqq 0$, or equivalently if $\|A^*f\|\leqq\|Af\|$ for all f in $\mathscr{H}$.

For a sequence β as above and a point ω in D, let

$$k^\beta_\omega(z)=\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}(\bar{\omega}z)^n.$$

Then the function k^β_ω is a point evaluation for $H^2(\beta)$; i.e., for f in $H^2(\beta)$,

$$(f, k^\beta_\omega)_\beta=f(\omega).$$

Note that $k^\beta_0=1$ (the function identically equal to 1), and that $C^*_\varphi k^\beta_\omega=k^\beta_{\varphi(\omega)}$.

Theorem 1. *If C_φ is hyponormal on the space $H^2(\beta)$, then* $\varphi(0)=0$.

Proof. Let C_φ be hyponormal on $H^2(\beta)$, and k^β_0 be point evaluation at 0. Then $\|C^*_\varphi f\|_\beta\leqq\|C_\varphi f\|_\beta$ for all f in $H^2(\beta)$, and if $f=k^\beta_0$ we have

$$\|C^*_\varphi k^\beta_0\|^2_\beta=\|k^\beta_{\varphi(0)}\|^2_\beta=\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}|\varphi(0)|^{2n}\leqq\|C_\varphi k^\beta_0\|^2_\beta=\|k^\beta_0\|^2_\beta=1,$$

which implies, since $\beta_0=1$, that $\varphi(0)=0$.

Theorem 1.5 in [2] states that if C^*_φ is hyponormal on H^2, then φ is univalent in D; the proof also applies to a general space $H^2(\beta)$. Also, by Theorem 1.4 in [2], if C^*_φ is hyponormal and not normal on H^2, then the Denjoy—Wolff α of φ (for definition and properties see [1]) is such that $|\alpha|=1$ and $\varphi'(\alpha)<1$. This result is not true in all spaces $H^2(\beta)$, as we can see from the following. Note that the spaces we are going to work with are "the small spaces $H^2(\beta)$" which consist of functions continuous on the closed unit disc. These spaces provide examples of some other interesting composition operators (for example, compact ones with no fixed point in the unit disc (see [8] and [10])).

First we need the following lemma.

Lemma 1. (Lemma 4.3 from [3].) *If A is hyponormal on $\mathscr{H}$, then for all $f\neq 0$ in $\mathscr{H}$, and for all $n>0$,*

$$\|A^nf\|\geqq\frac{\|Af\|^n}{\|f\|^{n-1}}.$$

Lemma 2. *Let the sequence β be such that $\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}<\infty$ and let C^*_φ be hyponormal on $H^2(\beta)$. Then* $\varphi(0)=0$.

Proof. By Lemma 1, for any $n \geqq 0$ we have

$$\|(C_\varphi^*)^n k_0^\beta\|_\beta^2 \geqq \frac{\|C_\varphi^* k_0^\beta\|_\beta^{2n}}{\|k_0^\beta\|_\beta^{2(n-1)}}.$$

But

$$\|C_\varphi^* k_0^\beta\|_\beta^2 = \|k_{\varphi(0)}^\beta\|_\beta^2 = \sum_{k=0}^\infty \frac{1}{\beta_k^2} |\varphi(0)|^{2k} = C_1^2$$

and

$$\|(C_\varphi^*)^n k_0^\beta\|_\beta^2 = \|k_{\varphi^{(n)}(0)}^2\| = \sum_{k=0}^\infty \frac{1}{\beta_k^2} |\varphi^{(n)}(0)|^{2k} \leqq \sum_{k=0}^\infty \frac{1}{\beta_k^2} = C_2^2$$

where $\varphi^{(n)}(0)$ is the n-th iteration of φ at 0.

We have that $\|k_0^\beta\|_\beta = 1$, and so

$$C_2^2 \geqq \|(C_\varphi^*)^n k_0^\beta\|_\beta^2 \geqq \|C_\varphi^* k_0^\beta\|_\beta^{2n} = C_1^{2n}.$$

If $C_1 > 1$, then $C_1^n \to \infty$, which is a contradiction with the previous inequality, and so $C_1 = 1$; i.e., $\varphi(0) = 0$.

Lemma 3. *If C_β^* is hyponormal on $H^2(\beta)$ and $\varphi(0)=0$, then $\varphi(z)=az$ where $|a| \leqq 1$.*

Proof. We use the idea of the proof of Theorem 2.4 in [7]. Let $\varphi(z) = a_1 z + {} + a_2 z^2 + a_3 z^3 + \dots$ and $f_k = \dfrac{z^k}{\beta_k}$. Then $\{f_k\}_{k=0}^\infty$ is an orthonormal basis for $H^2(\beta)$ and $\varphi(z) = \sum_{n=1}^\infty a_n \beta_n f_n$. Now

$$\|C_\varphi^* f_1\|_\beta^2 = \sum_{k=0}^\infty |(C_\varphi^* f_1, f_k)|^2 = \sum_{k=0}^\infty |(f_1, C_\varphi f_k)|^2 = \sum_{k=0}^\infty \frac{1}{\beta_k^2} |(f_1, \varphi^k)|^2 = \frac{1}{\beta_1^2} |a_1 \beta_1|^2.$$

Also

$$\|C_\varphi f_1\|_\beta^2 = \frac{1}{\beta_1^2} \|\varphi\|_\beta^2 = \frac{1}{\beta_1^2} \sum_{n=1}^\infty |a_n|^2 \beta_n^2.$$

The operator C_φ^* is hyponormal; i.e., for any f in $H^2(\beta)$, $\|C_\varphi f\|_\beta^2 \leqq \|C_\varphi^* f\|_\beta^2$. If $f = f_1$, we get that

$$\frac{1}{\beta_1^2} \sum_{n=1}^\infty |a_n|^2 \beta_n^2 \leqq \frac{1}{\beta_1^2} |a_1 \beta_1|^2$$

which implies that $0 = a_2 = a_3 = \dots$.

Theorem 2. *Let the sequence β be such that $\sum \dfrac{1}{\beta_n^2} < \infty$, and C_φ^* be hyponormal on $H^2(\beta)$. Then $\varphi(z) = az$, where $|a| \leqq 1$.*

Proof. The proof follows immediately from Lemma 2 and Lemma 3.

SCHWARTZ proved in [7] that a composition operator C_φ is normal on H^2 if and only if $\varphi(z)=az$ where $|a|\leqq 1$. Using the above results we can easily prove that the same statement holds for all spaces $H^2(\beta)$.

Theorem 3. *The operator C_φ is normal on $H^2(\beta)$ if and only if $\varphi(z)=az$ with $|a|\leqq 1$.*

Proof. It is trivial that if $\varphi(z)=az$, $|a|\leqq 1$, then C_φ is normal on $H^2(\beta)$. Conversely, if C_φ is normal, then C_φ is hyponormal and, by Theorem 1, $\varphi(0)=0$. But we also have C_φ^* hyponormal, and by Lemma 3, $\varphi(z)=az$ with $|a|\leqq 1$.

As a consequence of Theorem 2 and Theorem 3, we get an interesting example of family of spaces $H^2(\beta)$, where the only cohyponormal composition operators are the ones which are normal.

Corollary. *If $\sum 1/\beta_n^2<\infty$ and C_φ^* is hyponormal on $H^2(\beta)$, then C_φ^* is normal on $H^2(\beta)$.*

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